Crossing of Resonance in a Relativistic Electron Ring with Discrete and Continuous Eigenvalue Spectrum

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Abstract

The Vlasov formalism is used to calculate the first and second order perturbations of a ring of relativistic electrons after crossing the betatron resonance $\nu_r = 1$. The presence of coherent self-fields may result in separate crossing of the coherent and incoherent integral resonances associated with the discrete and continuous eigenvalues. The quantity χ giving the ratio of the spread in the single particle frequencies to the coherent frequency shift characterizes the excitation of the continuous spectrum, the eigenmodes of which are similar to the singular von Kampen modes in the theory of plasma oscillations. Compared with an existing approach the present derivation is valid for arbitrary values of χ and is selfconsistent in that it correctly includes the dynamics of the variable giving rise to the finite spread in ν_r. The increase in beam size resulting from crossing of the incoherent integral resonance is compared with that due to crossing of the half-integral resonance, for which an improved formula is derived with the Vlasov formalism.

1. Introduction

The resonance behaviour of a Ring of Relativistic Electrons in a regime where the radial single particle betatron frequency ν_r approaches unity and where coherent forces are considerable has been studied qualitatively [1] and in a quantitative model [2] with respect to its importance for electron ring accelerators (ERA) and possibly for electron storage rings.

It has been recognized that the formula for the increase in amplitude of a single particle crossing an integral resonance driven by a first harmonic magnetic field perturbation δB_z

(1) $$\delta = \left(\frac{\pi}{\Omega\sigma}\right)^{1/2} R\Omega^2 \frac{\delta B_z}{B_z}$$

(R orbit Radius, Ω revolution frequency, $\sigma = \Omega^2 \frac{d}{dt}\nu_r^2$) cannot hold if coherent fields arise during the resonance crossing. In ERA devices $\nu_r = 1$ is expected to occur after ring compression and an ion loading in excess of a few per cents. A quantitative approach [2] has shown that formula (1) applies for crossing of the coherent resonance which is separated from the incoherent resonance and is avoided in ERA by the inclusion of electric images on a "squirrel cage" or electric image cylinder. Crossing of the incoherent $\nu_r = 1$ (with ν_r including the shift due to stationary self-fields), however, gave a growth in beam dimension which was by a factor of $\frac{\Delta^2}{\Lambda^2}$ less than in (1), this factor giving the ratio of the spread in the square of the single particle frequency ν_r to the shift in the square of this frequency induced by the ions, i.e. the coherent frequency shift. This result was obtained in the limits $\frac{\Delta^2}{\Lambda^2} \ll 1$ and $\frac{\Delta^2}{\Lambda^2} \gg 1$ for a system of harmonic oscillators describing the radial motion with frequencies spreading about an average value and subject to a coherent force proportional to the c.o.m. displacement of the system from its unperturbed position.

The present study reconsiders this problem with two scopes:

1. derive results which hold for arbitrary $\chi \equiv \frac{\Delta[\Omega(1-\nu_1)]}{\Lambda^2/\Omega} \approx \frac{1}{2}\frac{\Delta^2}{\Lambda^2}$ carrying in mind that for ERA-applications χ may be of order 1 owing to the large energy spread required for stabilizing collective instabilities [3];

2. the existing approach [2] has analyzed the resonance response of a large number of particles oscillating harmonically in radial direction with spreading frequencies which were not consistent with the dynamics of the oscillators. In a real ring frequency spread may either come from (a) anharmonic oscillations or (b) a spread in the energy or canonical angular momentum P_θ of the particles. Hence, if (a) is faced, consistency requires the study of nonlinear oscillators, whereas in case (b) which is dominant in ERA, the dynamics of the $\theta - P_\theta$ motion should be included in the analysis. For this reason there is some question as to the validity of the purely radial harmonic oscillator model [2]. The present analysis claims consistency by employing a spread (b) along with the $\theta - P_\theta$ motion and is based on the Vlasov formalism.

We proceed in defining an appropriate equilibrium distribution function. Next we derive the inhomogeneous differential-integral equations governing the relevant momenta of the first and second order perturbations of this equilibrium. A singular eigenmode expansion is undertaken and the resulting integrals are evaluated using asymptotic methods. As a result of the crossing of resonance for the continuous spectrum of dipole oscillations the beam position remains unchanged due to phase mixing; the increase of the beam width is found roughly equal to the single-particle result for $\chi \gtrsim \frac{4}{3}$ and for $\chi \lesssim \frac{4}{3}$ it is reduced by the factor χ. The earlier formulae [2] for the limits $\chi \ll 1$ and $\chi \gg 1$ are thus confirmed by the present consistent theory.

2. Vlasov Model

The motion in phase space of a large number of electrons forming a relativistic electron ring is described by the Vlasov equation in cylindrical coordinates [4,5]

$$\frac{\partial f}{\partial t} + \frac{u_r}{\gamma}\frac{\partial f}{\partial r} + \frac{u_\theta}{r\gamma}\frac{\partial f}{\partial \theta} + \frac{u_z}{\gamma}\frac{\partial f}{\partial z} + \left[\frac{u_\theta^2}{r\gamma} + \frac{e}{m}\left(E_r + \frac{u_\theta}{\gamma}B_z - \frac{u_z}{\gamma}B_\theta\right)\right]\frac{\partial f}{\partial u_r}$$

(2)

$$+\left[-\frac{u_\theta u_r}{r\gamma} + \frac{e}{m}\left(E_\theta - \frac{u_r}{\gamma}B_z + \frac{u_z}{\gamma}B_r\right)\right]\frac{\partial f}{\partial u_\theta} + \frac{e}{m}\left(E_z - \frac{u_\theta}{\gamma}B_r + \frac{u_r}{\gamma}B_\theta\right)\frac{\partial f}{\partial u_z} = 0$$

with $u_r = \gamma \dot{r}$, $u_\theta = \gamma \dot{\theta} r$, $u_z = \gamma \dot{z}$, $\gamma^2 = 1 + u_r^2 + u_\theta^2 + u_z^2$ and electric and magnetic fields satisfying the Maxwell equations with appropriate boundary conditions and sources given by the charge and current densities

(3)
$$n(r,\theta,z,t) = e\iiint f \, du_r \, du_\theta \, du_z$$
$$\underline{j}(r,\theta,z,t) = e\int \frac{\underline{u}}{\gamma} f \, du_r \, du_\theta \, du_z$$

The unperturbed equilibrium is described by a stationary solution of (2) in terms of the total energy H^o and the canonical angular momentum P_θ as constants of the motion. If we expand the energy about its value for a purely orbital motion $r \equiv R(P_\theta)$, we obtain [5]

(4)
$$f^o = f^o(P_\theta, H^o_{\perp r} + H^o_{\perp z})$$
$$P_\theta \equiv m r u_\theta + e r A^o_\theta(r,z)$$
$$H^o_{\perp r} + H^o_{\perp z} \equiv \frac{m}{2\gamma_0}\left[u_r^2 + u_z^2 + \gamma_0^2 \Omega^2 (\nu_r^2 x^2 + \nu_z^2 z^2)\right]$$
$$x \equiv r - R(P_\theta)$$
$$\Omega(P_\theta) \equiv \frac{v_\theta}{R}, \quad v_0(P_\theta) \equiv \frac{u_{0\theta}}{\gamma_0}, \quad \gamma_0(P_\theta) \equiv 1 + u_{0\theta}^2$$
$$\nu_r^2(P_\theta) \equiv 1 + \frac{E^o_r + R(E^{o\prime}_r + v_0 B^{o\prime}_z)}{E^o_r + v_0 B^o_z} + \left(\frac{e E^o_r R}{m \gamma_0^3 v_0^2}\right)^2$$
$$\nu_z^2(P_\theta) \equiv \frac{R(E^{o\prime}_z - v_0 B^{o\prime}_r)}{E^o_r + v_0 B^o_z}$$

Next we assume a very slow time dependence of $\nu_r^2(P_\theta, t)$ such that the equilibrium (4) is only changing adiabatically and we may replace $H^o_{\perp r}$ in f^o by the adiabatic invariant $H^o_{\perp r}/\nu_r$. This variation in ν_r may be induced by slow accumulation of ion space charge.

With a small magnetic field perturbation δB_z we solve (2) using an expansion

$$f = f^{\circ} + \varepsilon f^{(1)} + \varepsilon f^{(2)} \tag{5}$$

We assume that the coherent force only depends on the total dipole moment of the beam and is not affected by higher order deformations. From the first order equation for $f^{(1)}$ we proceed to moment equations through integration over the transverse phase space. For the dipole moment induced by an ℓ-th harmonic $\delta B_z \sim e^{i\ell\theta}$

$$\begin{aligned} &X(u,t,\theta) \equiv \int x f^{(1)} d\sigma \\ &u \equiv P_\theta - \langle P_\theta \rangle \quad , \qquad d\sigma \equiv dr\, dz\, du_r\, du_z \end{aligned} \tag{6}$$

we find a differential-integral equation (appendix I)

$$\frac{\partial^2 X}{\partial t^2} + 2i\ell\Omega \frac{\partial X}{\partial t} - \Omega^2(\ell^2 - \nu_r^2) X = \Lambda^2 F^{\circ} \int X\, du + \frac{e v_0}{m\gamma_0} \delta B_z F^{\circ} \tag{7}$$

with

$$F^{\circ}(u) \equiv \int f^{\circ} d\sigma \tag{8}$$

Λ^2 includes the electron-electron, image-electron and ion-electron coherent force. Since the electron coherent displacement varies slowly compared with the ion bounce frequency, ions are trapped in the electron ring and their coherent force contribution is proportional to the electron coherent displacement (appendix I). Higher order momenta of $f^{(1)}$ are far away from resonance and may be disregarded here.

The second order equation is again transformed into moment equations and one sees that the resonance $\nu_r = \ell$ only affects upon the quadrupole moment

$$D(u,t,\theta) \equiv \int x^2 f^{(2)} d\sigma \tag{9}$$

which is governed by the third order partial differential equation (I.8) with nonlinear inhomogeneity. Its solution requires use of real quantitites for the field error and for X

(10) $$\overline{\delta B_z} \equiv \delta B_z \cos(\ell\theta) \; , \qquad \overline{X} \equiv \mathrm{Re}\left[X e^{i\ell\theta}\right]$$

which leads to the result (appendix I)

(11) $$D = \frac{\overline{X}^2}{F^0}$$

Hence, the problem of finding solutions for the first and second order perturbations has been reduced to solving equ. (7).

3. Normal Mode Analysis

The integral in (7) representing the coherent force introduces coupling between groups of particles with different values of P_θ . We proceed to solve (7) by expanding X in terms of the eigenmodes of the following auxiliary homogeneous equation

(12) $$\frac{\partial}{\partial t}\xi + i k_1 u \xi = -i k_1 G(u) \int_{-\infty}^{\infty} \xi(u', t)\, du'$$

where we assumed that $\Omega(l^2 - \nu_r^2)$ varies linearly with the canonical angular momentum u

(13) $$k \equiv k_0 + k_1 u \equiv \frac{\Omega(\ell^2 - \nu_r^2)}{2\ell}$$
$$G(u) \equiv \frac{\Lambda^2}{2\Omega \ell k_1} F^0$$

The ansatz

(14) $$\xi = \xi_\nu(u)\, e^{-i\nu k_1 t}$$

leads to an integral equation, which is well-known from the theory of linear plasma oscillations [6)]

(15) $$(u-\nu)\,\xi_\nu(u) = -G(u) \int_{-\infty}^{\infty} \xi_\nu(u')\, du'$$

The integral operator is unbounded and not self-adjoint but its eigenfunctions (distributions) are well understood and have completeness and orthogonality properties. There exists [6] a continuous spectrum of real eigenvalues ν corresponding to δ -function excitations at $u = \nu$, which are equivalent to the van Kampen modes in linear plasma oscillation theory

(16) $$\xi_\nu(u) = -\mathcal{P}\,\frac{G(u)}{u-\nu} + \lambda(\nu)\,\delta(u-\nu)$$

Here $\mathcal{P}$ signifies the principal value when this expression is integrated. $\lambda(\nu)$ is determined by the normalization condition

(17) $$\int_{-\infty}^{\infty} \xi_\nu(u)\,du = 1$$

and results as

(18) $$\lambda(\nu) = 1 + \mathcal{P}\int_{-\infty}^{\infty} \frac{G(u)}{u-\nu}\,du$$

In addition to the continuum of ν there exists a finite set of values ν_i with $\lambda(\nu_i) = 0$. These discrete eigenvalues are then determined by the dispersion relation

(19) $$1 + \int_{-\infty}^{\infty} \frac{G(u)}{u-\nu_i}\,du = 0$$

where ν_i is real if $G(\nu_i) = 0$, otherwise ν_i complex. The corresponding eigenmodes are

(20) $$\xi_{\nu_i}(u) = -\frac{G(u)}{u-\nu_i}$$

One can easily show that in the present problem with $G(u) \sim F^o(u) \gtrsim 0$ the only discrete eigenvalues are real and therefore correspond to stationary (undamped) coherent waves. In addition, it is necessary for such a ν_i to exist that the coherent selfforce ($\sim \Lambda^2$) is strong enough. For the remainder of this paper we specify

(21) $$F^o(u) = \begin{cases} \frac{3}{2d}\left[1-\left(\frac{u}{d/2}\right)^2\right] & |u| \le d/2 \\ 0 & |u| > d/2 \end{cases}$$

and take Δu as the width at half maximum, hence

(22) $$\Delta u \approx \frac{2}{3}d$$

This choice is more realistic than a Maxwellian or Lorentzian distribution with infinitely long tails. It is convenient to introduce the parameter

$$(23) \quad \chi \equiv \left| \frac{k_1 \langle \Omega \rangle \ell \, \Delta u}{\Lambda^2} \right| = \frac{\Delta S}{\Lambda^2 \langle \Omega \rangle}$$

which gives the ratio of the spread in the quantitiy

$$(24) \quad S(u) \equiv \Omega \left(\ell - \nu_1 \right)$$

and the coherent frequency shift Λ^2. For $\chi \lesssim 1$ there exists an undamped coherent oscillation ξ_o according to (20) with eigenvalue ν_o. This mode disappears for $\chi > 1$ and the remaining modes are the continuous eigenmodes (16), which show phase mixing or Landau damping. For $\chi \gg 1$ we have essentially single particle behaviour with no coherent effects.

The existence of a real discrete eigenvalue ν_o is pertinent to distribution functions $F^o(u)$ which are nonzero for only a finite and sufficiently small range of u. This eigenvalue is associated with an undamped dipole oscillation and is predicted in ERA-devices for most of the practicable boundary conditions [5]. It may require extremely large energy spread for damping in the linear regime of the resisitive [8] and electron-ion instability [9].

The set of solutions (16), (17) is complete for a sufficiently large class of functions (which obey, for instance, a Hölder condition [6]) defined on $-\infty < u < \infty$ and it is orthogonal on the solutions of the adjoint equation [6]

$$(25) \quad (u - \nu) \tilde{\xi}_\nu (u) = - \int_{-\infty}^{\infty} G(u') \, \xi_\nu (u') \, du'$$

Hence, in order to solve (7), we use the expansion

$$(16) \quad X(u,t) \equiv X_{coh}(u,t) + X_{inc}(u,t) \equiv a_o(t) \, \xi_o(u) + \int_{-\infty}^{\infty} A_\nu(t) \, \xi_\nu(u) \, d\nu$$

4. Resonance Crossing

For slowly time-varying ν_r^2 the driving term in the inhomogenous equation (7) excites those eigenmodes of the corresponding homogeneous equation (12) which are in resonance. If we assume linear time dependence in k_o

$$k_0 = g_0 - g_1(t-t_0) \tag{27}$$

and use the orthogonality properties with the adjoint eigenmodes we find for the amplitude function of the coherent stationary wave the equation

$$\frac{-i}{2\Omega l}\frac{\partial^2 a_0}{\partial t^2} + \frac{\partial a_0}{\partial t} + i\left(k_0 + k_1\nu_0\right)a_0 = h_0 \equiv -2i\frac{ev_0}{m\gamma_0}\delta B_z \frac{k_1^2\Omega l}{\Lambda^4\int\frac{F^{\circ\prime}(u)}{(u-\nu_0)^2}du} \tag{28}$$

and for waves of the continuous spectrum

$$\frac{-i}{2\Omega l}\frac{\partial^2 A_\nu}{\partial t^2} + \frac{\partial A_\nu}{\partial t} + i\left(k_0 + k_1\nu\right)A_\nu = H_\nu \tag{29}$$

with

$$H_\nu \equiv 2i\frac{ev_0}{m\gamma_0}\delta B_z \frac{k_1^2\Omega l F^{\circ}(\nu)}{4k_1^2\Omega^2 l^2\lambda^2(\nu) + \pi^2\Lambda^4 F^{\circ 2}(\nu)} \; , \qquad \lambda(\nu) \equiv 1 + \frac{\Lambda^2}{2k_1\Omega l}\mathcal{P}\int\frac{F^{\circ}(u)}{u-\nu}du \tag{30}$$

if $F^{\circ}(\nu) \neq 0$, otherwise $H_\nu = 0$

Since in (28), (29) the coefficients of the terms a_o, A_ν are sufficiently close to zero, the time independence of the right sides permits to describe a particular solution of (28), (29) by the reduced equation

$$\frac{\partial y}{\partial t} + i\,g(t)\,y = b \tag{31}$$

With the initial condition $y(t_0) = 0$ we get

$$y(t) = b\exp\left[-i\int_{t_0}^{t} g(t')\,dt'\right]\int_{t_0}^{t}\exp\left[i\int_{t_0}^{t'} g(t'')\,dt''\right]dt' \tag{32}$$

(a) Crossing of resonance for the coherent stationary wave

Inserting (27) into (28) we assume that $g \equiv g_0 + k_1\nu_0 - g_1(t-t_0)$ crosses zero. This coherent resonance results in c.o.m. (i.e. dipole-) oscillations of the beam. To evaluate (32) we write

$$(33) \quad a_0(t) = h_0\left(\frac{2}{g_1}\right)^{1/2} \exp\left\{ i \, \frac{\left[g_0 + k_1\nu_0 - g_1(t-t_0)\right]^2}{2g_1} \right\} \int_{\eta_0}^{\eta_0 + \left(\frac{g_1}{2}\right)^{1/2}(t-t_0)} e^{-i\eta^2} d\eta$$

and find for small g_1

$$(34) \quad a_0(t) \approx h_0\left(\frac{\pi}{g_1}\right)^{1/2}(1-i) \exp\left\{ i \, \frac{\left[g_0 + k_1\nu_0 - g_1(t-t_0)\right]^2}{2g_1} \right\}$$

where we have assumed that the integration limits obey

$$(35) \quad \begin{aligned} \eta_0 &\equiv - \frac{g_0 + k_1\nu_0}{(2g_1)^{1/2}} \ll -1 \\ \eta_0 &+ \left(\frac{g_1}{2}\right)^{1/2}(t-t_0) \gg 1 \end{aligned}$$

and therefore can be shifted towards infinity [2).]

Since g_1 gives the rate of crossing through the resonance, we observe that the amplitude factor in (34) has the same structure as for a single particle crossing the resonance, except that h_0 has to be evaluated according to (28) for a given distribution of particles.

(b) Crossing of resonance for the continuous spectrum ("incoherent" resonance)

We have to solve (29) for each ν with $F^0(\nu) \neq 0$ and $g(\nu,t) \equiv g_0 + k_1\nu - g_1(t-t_0)$ crossing zero. Under the same conditions as in (35), with ν_0 replaced by ν, we obtain

$$(36) \quad A_\nu(t) \approx H_\nu\left(\frac{\pi}{g_1}\right)^{1/2}(1-i) \exp\left\{ i \, \frac{\left[g_0 + k_1\nu - g_1(t-t_0)\right]^2}{2g_1} \right\}$$

and with (16)

$$(37)\quad X_{inc}(u,t) = \left(\frac{\pi}{g_1}\right)^{1/2}(1-i)\int_{-\infty}^{\infty} H_\nu \exp\left\{i\,\frac{[g_0 + k_1\nu - g_1(t-t_0)]^2}{2g_1}\right\}\left[-\beta\frac{G(u)}{u-\nu} + \lambda(\nu)\,\delta(u-\nu)\right]d\nu$$

The principal value integral in (37) with the rapidly oscillating exponential function can be evaluated asymptotically for $g_1 \to 0$ with the result that after crossing the resonance the leading contribution to the integral is from the residue at $\nu = u$ (App.II.4). Hence, with (10, (30) and $\frac{e v_0 B_z}{m\gamma_0} = -R\Omega^2$

$$(38)\quad X_{inc}(u,t) = \left(\frac{\pi}{g_1}\right)^{1/2} R\Omega^2\,\frac{\delta B_z}{B_z}\,\frac{k_1}{\Lambda^2}\,i(1-i)\,G(u)\,\frac{\lambda - \pi i G}{\lambda^2 + \pi^2 G^2}\exp\left\{i\,\frac{[g_0 + k_1 u - g_1(t-t_0)]^2}{2g_1}\right\}$$

5. Ring Displacement and Radial Growth After Resonance Crossing

From the above derived expressions for the dipole excitation we obtain by integrations over the angular momentum the radial displacement and grwoth in radial beam size. Since we have in general, the system enters into the "incoherent" resonance first and then may, eventually, cross the coherent resonance. In ERA applications, however, the coherent resonance is not crossed and there remains only an effect by the proximity of this resonance.

(a) Radial displacement

The perturbation of the radial position $\langle r\rangle^{(1)}$ is given by (App.I.4) and requires integration of $X_{coh}(u,t)$ and $X_{inc}(u,t)$. The incoherent part, induced by the continuous spectrum, has a vanishing contribution to $\langle r\rangle^{(1)}$. This results from the fact that due to phase incoherence the integral arising from (38) has the asymptotic limit zero for $g_1 \to 0$ (App.II.3).

If the system stays in the proximity of the coherent resonance without crossing it, there results a stationary amplitude given by a particular solution of (24)

$$(39)\quad a_0 = \frac{h_0}{i(k_0 + k_1\nu_0)}$$

where h_o follows from (28) and ν_o from the dispersion integral (19). In the limit of no spread, i.e. $F^o = \delta(u)$, we find

$$(40) \qquad \nu_o = \frac{\Lambda^2}{2\langle\Omega\rangle k_1 l} \,, \qquad h_o = i\,\frac{\langle R\Omega\rangle \frac{\delta B_z}{B_z}}{2l}$$

and thus

$$(41) \qquad \langle r\rangle^{(1)} = a_o = \frac{\langle R\Omega\rangle \frac{\delta B_z}{B_z}}{2l\omega}$$

with the coherent frequency given in this case by

$$(42) \qquad \omega = \frac{\langle \Omega^2(l^2-\nu_r^2)\rangle + \Lambda^2}{2l\langle\Omega\rangle}$$

For finite χ, but $\chi < 1$, we find that h_o is slightly modified and vanishes for $\chi = 1$.

In case of crossing of the coherent resonance at a rate

$$(43) \qquad \sigma = \langle\Omega^2\rangle \frac{d}{dt}(\nu_r^2)$$

(41) is replaced by

$$(44) \qquad \langle r\rangle^{(1)} \approx \left(\frac{\pi}{l\langle\Omega\rangle\sigma}\right)^{1/2} \langle R\Omega^2\rangle \frac{\delta B_z}{B_z}$$

which agrees with the single-particle formula.

(b) Radial beam width

We evaluate the formula (App.I.11) for the perturbation of the radial beam size after crossing the "incoherent" resonance. Defining the real quantity

$$(45) \qquad \bar{X}(u,t,\theta) \equiv \mathrm{Re}\left[(X_{coh} + X_{inc})e^{il\theta}\right]$$

we have

$$(46) \qquad \int \frac{\bar{X}^2}{F^o}du = \int \frac{\left[\mathrm{Re}(X_{coh}e^{il\theta})\right]^2}{F^o}du + 2\int \frac{\mathrm{Re}(X_{coh}e^{il\theta})\mathrm{Re}(X_{inc}e^{il\theta})}{F^o}du + \int \frac{\left[\mathrm{Re}(X_{inc}e^{il\theta})\right]^2}{F^o}du$$

With the exponential in (38) the second integral on the right side vanishes by phase mixing (App.II). Here we assume that the first integral containing the contribution due to the proximity of the coherent resonance is small, and obtain with the third integral

(47) $$\int \frac{\bar{X}^2}{F^0}\,du \approx \frac{1}{2}\int \frac{|X_{inc}|^2}{F^0}\,du$$

Evaluating numerically (30) with the distribution function (21) we get with (37), (47) and (App.I.10)

(48) $$\langle \rho^2\rangle^{(2)} \approx \left[\left(\frac{\pi}{\ell\langle\Omega\rangle\sigma}\right)^{1/2}\langle R\Omega^2\rangle\,\frac{\delta B_z}{B_z}\right]^2 \phi(\chi)$$

The reduction factor results as

(49) $$\phi(\chi) = \frac{9}{4}\chi^2\int_{-1}^{1}\frac{1-y^2}{\frac{3}{2}\pi^2(1-y^2)^2+4\left\{\chi-\left[y-\frac{1}{2}(1-y^2)\ln\frac{1-y}{1+y}\right]\right\}^2}\,dy$$

and with sufficient accuracy we find

(50) $$\phi(\chi) \approx \begin{cases} \chi^2/4 & \chi \lesssim \frac{4}{3} \\ 1/2 & \chi > \frac{4}{3} \end{cases}$$

This scaling is in agreement with the result of the calculation [2], which was done for the limits $\chi \ll 1$ and $\chi \gg 1$.

6. The Half-integral Resonance

It is of interest to compare (48) with the beam widening that results from the simultaneously occuring half-integral resonance $2\nu_r = 2$. This resonance was found [10] to be driven by a second harmonic field gradient error on the basis of an analysis of the radial equation of motion which was decoupled from azimuthal motion. A more careful analysis shows that this resonance is also driven by a second harmonic field error, which acts as an azimuthal force and results in a radial perturbation by virtue of the centrifugal force term. The differential equation governing the quadratic moment of the first order perturbation is readily obtained from (App.I.1) and is simplified by the absence of coherent forces

$$(51)\qquad L^* \int x^2 f^{(1)} d\sigma = \frac{i}{2}\frac{e}{\gamma_0}\left[\frac{1}{m}\int \frac{u_\theta}{\gamma}\,\delta\left(\frac{\partial B_z}{\partial r}\right)^{(\ell=2)} x^2 f^0 d\sigma + \int \frac{u_r^2}{\gamma}\left(\delta B_z\right)^{(\ell=2)} r\, x \frac{\partial f^0}{\partial P_\theta} d\sigma\right]$$

with L^* the same differential operator as in the equation for $\int x^2 f^{(2)} d\sigma$ (App.I.8). No ring displacements occur after crossing the half-integral resonance, whereas the square of the beam width is increased according to

$$(52)\qquad \langle \rho^2\rangle^{(1)} \approx \left(\frac{\pi}{2\langle\Omega\rangle\sigma}\right)^{1/2} \langle\rho^2\rangle^0 \langle\Omega^2\rangle \left[\frac{\delta B_z}{B_z} + \frac{\langle R\rangle}{B_z}\delta\left(\frac{\partial B_z}{\partial r}\right)\right]^{(\ell=2)}$$

7. Tolerable Field Errors

From the requirement of a resonant change in ring quality small compared with the orginal radial beam size ρ_0 we get the following tolerances for first harmonic errors

$$(53)\qquad \frac{\delta B_z}{B_z}^{(\ell=1)} \ll \frac{\rho_0}{R}\left(\frac{\sigma}{\Omega^3\pi}\right)^{1/2} \phi(\chi)^{-1/2} \qquad \text{incoherent resonance } \nu_r = 1$$

$$(54)\qquad \frac{\delta B_z}{B_z}^{(\ell=1)} \ll \frac{\rho_0}{R}\frac{2\omega}{\Omega} \qquad \text{proximity of coherent resonance}$$

$$(55)\qquad \frac{\delta B_z}{B_z}^{(\ell=1)} \ll \frac{\rho_0}{R}\left(\frac{\sigma}{\Omega^3\pi}\right)^{1/2} \qquad \text{coherent resonance (not in ERA)}$$

and for a second harmonic error according to (52)

$$(56)\qquad \left[\frac{\delta B_z}{B_z} + \frac{R}{B_z}\delta\left(\frac{\partial B_z}{\partial r}\right)\right]^{(\ell=2)} \ll \left(\frac{\sigma}{2\Omega^3\pi}\right)^{1/2}$$

As an example we take the case of the Garching "Pustarex" ERA [11] with parameters for the compressed ring $\Omega \approx 10^{10}$ sec^{-1}, $\rho_0/R \approx 10^{-1}$ $\omega/\Omega \approx .02$ and $\sigma/\Omega^3 \approx 2\cdot 10^{-8}$, which corresponds to a change of ν_r^2 by .02 in 1 msec. This may result from an effective increase of the ion loading $f \equiv \frac{ZN_i}{N_e}$ up to .035 leading to resonance by stripping heavy ions to high Z. If we assume $\chi \approx \frac{1}{4}$ [5] we get from (53)

$$\left(\frac{\delta B_z}{B_z}\right)^{(\ell=1)} \ll 2\cdot 10^{-5} \tag{57}$$

and from (56) also

$$\left[\frac{\delta B_z}{B_z} + \frac{R}{B_z}\,\delta\left(\frac{\partial B_z}{\partial r}\right)\right]^{\ell=2} \ll 2\cdot 10^{-5} \tag{58}$$

With these strong requirements on the magnetic field it appears necessary in this case to avoid crossing of $\nu_r = 1$ by keeping f below the critical limit of a few percent or increase substantially the crossing speed σ by a rapidly changing field index n.

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Appendix I

Moment equations derived from the Vlasov equation.

1. First order moment

From (2), (5) we obtain with the independent variables

$r\,,\ \theta\,,\ z\,,\ u_r\,,\ P_\theta\,,\ u_z\,,\ t$

(I.1)
$$Lf^{(1)} \equiv \frac{\partial f^{(1)}}{\partial t} + \frac{u_\theta}{r\gamma}\frac{\partial f^{(1)}}{\partial \theta} + \frac{u_r}{\gamma}\frac{\partial f^{(1)}}{\partial r} + \frac{u_z}{\gamma}\frac{\partial f^{(1)}}{\partial z} + \left[\frac{u_\theta^2}{r\gamma} + \frac{e}{m}\left(E_r^0 + \frac{u_\theta}{\gamma}B_z^0\right)\right]\frac{\partial f^{(1)}}{\partial u_r} + \frac{e}{m}\left[E_z^0 - \frac{u_\theta}{\gamma}B_r^0\right]\frac{\partial f^{(1)}}{\partial u_z}$$
$$= -\frac{e}{m}\left[E_r^{(1)} + \frac{u_\theta}{\gamma}\left(B_z^{(1)} + \delta B_z\right) - \frac{u_z}{\gamma}B_\theta^{(1)}\right]\frac{\partial f^0}{\partial u_r} - e\left[E_\theta^{(1)} - \frac{u_r}{\gamma}\left(B_z^{(1)} + \delta B_z\right) + \frac{u_z}{\gamma}B_r^{(1)}\right] r\frac{\partial f^0}{\partial P_\theta} - \frac{e}{m}\left[E_z^{(1)} - \frac{u_\theta}{\gamma}B_r^{(1)} + \frac{u_r}{\gamma}B_\theta^{(1)}\right]\frac{\partial f^0}{\partial u_z}$$

In the highly relativistic limit the self-field perturbations only depend on the ring displacement

(I.2)
$$\langle r\rangle^{(1)} \equiv \frac{\int r f\, d\sigma\, du}{\int f\, d\sigma\, du} - \langle R\rangle = \int X\, du + \int\left[(R - \langle R\rangle)\int f^{(1)} d\sigma\right] du$$

using (6), (8) and

$$du = dP_\theta\,,\quad \langle R\rangle = \int R F^0 du\,,\quad \int F^0 du = 1$$

With appropriate integrations over the transverse phase space equ. (I.1) leads to the first order equation for the dipole moment $X(u,t,\theta)$

(I.3)
$$\frac{\partial^2}{\partial t^2}X + 2\Omega\frac{\partial^2}{\partial\theta\,\partial t}X + \Omega^2\left(\nu_r^2 + \frac{\partial^2}{\partial\theta^2}\right)X =$$
$$-eRR'F^0\left(\frac{\partial}{\partial t} + \Omega\frac{\partial}{\partial\theta}\right)E_\theta^{(1)} + \frac{e}{m\gamma_0}F^0\left[E_r^{(1)} + v_0\left(B_z^{(1)} + \delta B_z\right)\right]$$

For perturbations $\sim e^{il\theta}$ and slow time-dependence we get from (I.1)

$$\int f^{(1)} d\sigma \approx \frac{X}{R} + \frac{ie}{l\Omega}\frac{d}{du}\left(E_\theta^{(1)} R F^0\right)$$

and for $\dfrac{R - \langle R\rangle}{R} \ll 1$ we may approximate (I.2) by

(I.4)
$$\langle r\rangle^{(1)} \approx \int X\, du$$

We obtain equ.(7) if we collect the coherent force contributions in a coefficient Λ^2 defined by

$$\text{(I.5)}\quad \Lambda^2 \langle r \rangle^{1} \equiv -ie\,l\,\Omega R R' E_\theta^{(1)} + \frac{e}{m\gamma_0}\left(E_r^{(1)} + v_0 B_z^{(1)}\right)$$

where Λ^2 includes the ion space charge shift for a fractional ionization $f = \frac{Z N_i}{N_e}$ and the coherent frequency shift by electrons and images [5])

$$\text{(I.6)}\quad \Lambda^2 \equiv 2\Omega \nu_r U \equiv \Omega^2 \mu\left[-2\left(\frac{R}{\varrho_0}\right)^2\left(\frac{1}{\gamma_0^2} - f\right) + \ln\left(\frac{8R}{\varrho}\right) + 4\,\frac{\varepsilon_e^{coh} - \beta^2 \varepsilon_m^{coh}}{(1-\zeta)^2}\right]$$

2. Second order moment

The second order perturbation $f^{(2)}$ obeys

$$Lf^{(2)} = -\frac{e}{m}\left[E_r^{(1)} + \frac{u_\theta}{\gamma}\left(B_z^{(1)} + \delta B_z\right) - \frac{u_z}{\gamma} B_\theta^{(1)}\right]\frac{\partial f^{(1)}}{\partial u_r} - e\left[E_\theta^{(1)} - \frac{u_r}{\gamma}\left(B_z^{(1)} + \delta B_z\right) + \frac{u_z}{\gamma} B_r^{(1)}\right] r \frac{\partial f^{(1)}}{\partial P_\theta}$$

$$\text{(I.7)}\qquad - \frac{e}{m}\left[E_z^{(1)} - \frac{u_\theta}{\gamma} B_r^{(1)} + \frac{u_r}{\gamma} B_\theta^{(1)}\right]\frac{\partial f^{(1)}}{\partial u_z}$$

Second order field perturbations may be discarded if we assume that only ring displacements, which are taken into account by the first order equation, may affect upon the coherent fields. Owing to the quadratic nature of the inhomogeneous terms, only the quadrupole moment $D(u,t,\theta) \equiv \int x^2 f^{(2)} d\sigma$ is resonant at $\nu_r = 1$. It is a solution of the partial differential equation

$$\text{(I.8)}\quad \frac{1}{8\Omega}\frac{\partial^3 D}{\partial t^3} + \frac{3}{8}\frac{\partial^3}{\partial t^2 \partial\theta} D + \frac{\Omega}{2}\left(\nu_r^2 + \frac{3}{4}\frac{\partial^2}{\partial\theta^2}\right)\frac{\partial D}{\partial t} + \frac{\Omega^2}{2}\left(\nu_r^2 + \frac{1}{4}\frac{\partial^2}{\partial\theta^2}\right)\frac{\partial D}{\partial\theta}$$
$$= \lambda_{u,t,\theta}\left(E_r^{(1)}, E_\theta^{(1)}, B_z^{(1)}, \delta B_z, \overline{X}\right)$$

where the nonlinear inhomogeneity $\lambda_{u,t,\theta}$ contains zero and second azimuthal harmonics and involves differentiations with respect to t, θ . We observe that real perturbations according to (10) have to be used in (I.8), because $\lambda_{u,t,\theta}$ is nonlinear and superposition no longer holds.

Observing equ.(I.3) along with the asumption that f^o depends quadratically on x, which is consistent with harmonic transverse betatron oscillations in equilibrium it can be verified that (I.8) is solved by (11).

To determine the radial beam width we calculate the averaged square amplitude

$$\text{(I.9)} \quad \langle \rho^2 \rangle \equiv \frac{\int (r - \langle r \rangle)^2 (f^\circ + f^{(1)} + f^{(2)})\, d\sigma\, du}{\int (f^\circ + f^{(1)} + f^{(2)})\, d\sigma\, du}$$

which gives, in leading order, with (11)

$$\text{(I.10)} \quad \langle \rho^2 \rangle = \langle \rho^2 \rangle + \int \left[\frac{\langle R \rangle^2 - \langle \rho^2 \rangle^\circ - R^2}{R} + 4(R - \langle R \rangle) \right] \bar{X}\, du + \int \frac{\bar{X}^2}{F^\circ}\, du$$

Appendix II

Asymptotic Integrals

The asymptotic value for $g_1 \to 0$ of an integral of the type

$$\text{(II.1)} \quad I(g_1) = \int_C V(z) \exp\left\{ i \frac{\psi(z)}{g_1} \right\} dz \quad ,$$

where C is a contour in the complex z-plane and $V(z)$, $\psi(z)$ are analytic in some region including C, is required. With the method of steepest descent (see for instance [7]) $I(g_1)$ is obtained by summing up contributions at the critical points of ψ (end points, stationary or saddle points) after deforming the contour to coincide with the lines of steepest descent at critical points. The stationary points of

$$\text{(II.2)} \quad \psi(z) \equiv \frac{1}{2} \left[g_0 + k_1 z - g_1 (t - t_0) \right]^2$$

occur at resonance $\left(\frac{d\psi}{dz} = 0 \right)$, thus for the values of (37) and $\int X_{inc}(u,t)\,du$ after resonance we need not take into account stationary points. End point contributions are found of order g_1 [7], thus

$$\text{(II.3)} \quad \int X_{inc}(u,t)\, du \sim g_1^{-1/2} \cdot O(g_1) \sim g_1^{1/2} \to 0$$

Evaluation of (37) requires consideration of the singularity at $\nu = u$. To this end we deform the integration contour (fig.1)

$$\mathcal{P}\int \frac{H_\nu}{u-\nu}\exp\left[i\,\frac{\psi(\nu)}{g_1}\right]d\nu = \frac{1}{2}\int_{C^+} + \frac{1}{2}\int_{C^-}$$

$$= \int_{C^+} + i\pi H_u \exp\left[i\,\frac{\psi(u)}{g_1}\right]$$

(II.4)

$$= \int_{D_1} - \int_{D_2} + i\pi H_u \exp\left[i\,\frac{\psi(u)}{g_1}\right] \approx i\pi H_u \exp\left[i\,\frac{\psi(u)}{g_1}\right]$$

and obtain the result that the leading contribution is due to the residuum at $\nu = u$.

Fig.1

Deformation of contour of integration in the complex ν-plane with critical points and lines of steepest descent